Trajectory Levels	Level Description
Counting	**To count a set of objects, children must learn to count verbally - saying the list of number names in order, as well as learning the system that generates new numbers names.** **Counting includes 1) the ability to say number words in correspondence with objects (enumerate objects), 2) understanding that the last number word said when counting refers to how many items have been counted (cardinality), and 3) using counting strategies to solve problems.**

Age Range

1	1	Counting: Number Word Sayer: Foundations	No verbal counting. Names some number words with no sequence.
1	2	Counting: Chanter	Chants number words in “sing-song” fashion and may run them together. The number words may be indistinguishable from one another (‘onetwothree’). May begin a nonverbal object “counting” such as copying an adult’s item-by-item placement of objects.
2	2	Counting: Reciter	Verbally counts with distinct words, not necessarily in the correct order above “five.” May say “One, two, three, four, five, seven.” If knows more number words than number of objects, rattles them off quickly at the end; if more objects, “recycles” number words (inflexible list exhaustion)
3	3	Counting: Reciter (10)	Verbally counts to ten with some correspondence with objects, but may either continue an overly rigid correspondence or exhibit performance errors (e.g., skipping, double-counting).
3	3	Counting: Corresponder	Keeps one-to-one correspondence between counting words and objects (one word for each object), at least for small groups of objects laid in a line.
4	4	Counting: Counter (Small Numbers)	Accurately counts objects in a line to 5 and answers the “how many” question with the last number counted, understanding that this represents the total number of objects (the cardinal principle).
4	4	Counting: Counter (10)	Counts arrangements of objects to 10 with understanding of the cardinal principle. May be able to read and write numerals to represent 1–10. May be able to tell the number just after or just before another number, but only by counting up from 1. Verbal counting to 20 is developing.
4	4	Counting: Producer (Small Numbers)	Counts out objects to 5. Recognizes that counting is relevant to situations in which a certain number must be placed. Produces a group of 4 objects.
5	5	Counting: Counter and Producer (10+)	Counts and counts out objects accurately to 10, then beyond (to about 30). Has explicit understanding of cardinality (how numbers tell how many). Keeps track of objects that have and have not been counted, even in different arrangements. Writes or draws to represent 1 to 10 (then, 20, then 30). Gives next number (usually to 20s or 30s). Separates the decade and the ones part of a number word and begins to relate each part of a number word/numeral to the quantity to which it refers. Recognizes errors in others’ counting and can eliminate most errors in own counting (point-object) if asked to try hard.

Trajectory	Levels		Level Description
5	5	Counting: Counter Backward from 10	Counts backward from 10 to 1, verbally, or when removing objects from a group. “10, 9, 8, 7, 6, 5, 4, 3, 2, 1!”
6	6	Counting: Counter from N (N + 1, N -1)	Counts verbally and with objects from numbers other than 1 (but does not yet keep track of the number of counts). Immediately determines numbers just after or just before.
6	6	Counting: Skip Counter by 10s to 100	Skip counts by tens up to 100 or beyond with understanding; e.g., “sees” groups of 10 within a quantity and counts those groups by 10 (this relates to multiplication and algebraic thinking). “10, 20, 30 … 100.”
6	6	Counting: Counter to 100	Counts to 100. Makes decade transitions (e.g., from 29 to 30) starting at any number. “… 78, 79 … 80, 81 …”
6	6	Counting: Counter on Using Patterns	Keeps track of counting acts, but only by using numerical patterns (spatial, auditory, or rhythmic) for adding 1 to about 3.
6	6	Counting: Skip Counter	Counts by fives and twos with understanding.
6	6	Counting: Counter on Keeping Track	Counts forward or back from a given number keeping track of counting acts numerically, first using objects, then by “counting counts.”
6	6	Counting: Counter of Quantitative Units/Place Value	Understands the base ten numeration system and place value concepts, including ideas of counting in units and multiples of hundreds, tens, and ones. When counting groups of ten, can decompose into 10 ones if that is useful. Understands value of a digit according to the place of the digit within a number. Counts unusual units, such as “wholes” when shown combinations of wholes and parts.
6	6	Counting: Counter Beyond 100	Accurately counts beyond 100, recognizing the patterns of ones, tens, and hundreds.
7	7	Counting: Number Conserver	Consistently conserves number (i.e., believes number has been unchanged), even in face of perceptual distractions such as the spreading out of objects in a collection.
7	7	Counting: Counter Forward and Back	Counts “counting words” (single sequence or skip counts) in either direction. Recognizes that decades sequence mirrors single digit sequence. Switches between sequence and composition views of multidigit numbers easily.

Learning Trajectories

Trajectory Levels	Level Description
Subitizing	**Subitizing is quickly recognizing and naming the number in a group without counting. “Subitize” comes from the Latin “to arrive suddenly.” It begins in infancy with a sensitivity to number. Then children learn to recognize very small numbers. Later, they learn to do it quickly-- perceptual subitizing.. Another qualitative progression is their ability to see several groups and combine them quickly into one quantity - conceptual subitizing.**

Age Range			
0	1	Subitizing: Number Senser	Has inborn specific "sensers" for number from the first months of life without explicit knowledge of number. Intuitively distinguishes between groups of 1 and of 2 (and possibly 2 and 3). Also shows sensitivity to ratios of quite large numbers. (Approximate Number System, or ANS). These are pre-math, foundational abilities.
1	2	Subitizing: Very Small Number Recognizer	Begins connecting small quantities to number words to form an explicit idea of cardinality, or “how-many-ness.” Following the child's first birthday, the number words “one” and “two” are often learned. Other general terms such as “more” and "less" usually follow. Only over time do they begin to understand that all groups labelled with the same number word have the same amount.
2	3	Subitizing: Maker of Small Collections	Makes a small collection (usually 1–2 and possibly 3) with the same number as another collection (via mental model; i.e., not necessarily by matching—for that process, see Compare Number). Might also be verbal but often is not. May not recognize spatial structures at first, and may count this.
2	3	Subitizing: Small Collection Namer	Names groups of 1, 2, and 3 with increasing accuracy. Most children of about 34-39 months of age can accurately name groups of 1, 2, and 3. Many children learn to recognize and name groups of 4 about 6 months later. For a Maker of Small Collections (the previous level), the child may rely on matching strategies to make their small collection. In Small Collection Namer, the child is actually able to recognize small groups without relying on a model or matching strategy.
3	4	Subitizing: Perceptual Subitizer to 4	Instantly recognizes collections up to 4 briefly shown and verbally names the number of items.
4	4	Subitizing: Perceptual Subitizer to 5	Instantly recognizes briefly shown collections up to 5 and verbally names the number of items. Recognizes and uses spatial and numeric structures beyond the situations in which they were already experienced (i.e., in which they were initially learned).
4	4	Subitizing: Conceptual Subitizer to 5	Verbally labels all arrangements to about 5, shown only briefly, by seeing the parts and quickly knowing the whole. Conceptual subitizing refers to the ability of children to identify a whole quantity as a result of composing smaller quantities (recognized through perceptual subitizing) that make up the whole.
4	6	Subitizing: Conceptual Subitizer to 7	Verbally labels all arrangements to 6, then 7, when shown only briefly.

Trajectory	Levels		Level Description
5	6	Subitizing: Conceptual Subitizer to 10	erbally labels most briefly shown arrangements of all numbers 2 to 10. Children may know some familiar ones (“5 and 5 make 10” is common) early, but this level is reached when most all combinations of all numbers up to 10 are recognized (e.g., 7 and 2 seen as 9; 5 and 3 seen as 8; etc.). Uses structures such as tens-frames to recognize larger quantities.
6	7	Subitizing: Conceptual Subitizer to 20	Verbally labels structured arrangements of 10 to 20, shown only briefly, by seeing the parts and quickly knowing the whole. Spontaneously makes use of a top-down strategy to subitizing large quantities (Nes, 2009). Children may know some familiar ones (“10 and 10 make 20” is common) early, but this level is reached when most all combinations of numbers from 1 to 10 are recognized (e.g., 7 and 9 is seen as 16).
7	7	Subitizing: Conceptual Subitizer with Place Value	Verbally labels structured arrangements, shown only briefly, using groups, skip counting, and place value.
8	8	Subitizing: Conceptual Subitizer with Place Value and Multiplicative Thinking	Verbally labels structured arrangements, shown only briefly, using groups, multiplicative thinking, and place value. This level builds on the previous level, such that children are able to use the base-10 system to conceptually subitize larger numbers. Children are able to verbalize the quantity of 10's they see.

Trajectory Levels	Level Description
Comparing Number	**Children learn to order numbers (determining which of two numbers is "larger than" the other, or sequencing numerals or sets).**

Age Range

0	1	Comparing Number: Comparison Senser: Foundations	From the first months of life, children are sensitive to a change in the number, either of a change of very small collections, such as 1 vs. 2, or large changes in larger collection, such as double the number. Therefore, we know infants have an unconscious, innate sensitivity to such simple equivalence comparisons.
1	3	Comparing Number: Early Comparison Corresponder	Many-to-One Corresponder: Consciously recognizes that two very small collections have the “same number” by intuitively making a correspondence between the items in each collection. At this level, in certain situations, children may also put objects, words, or actions in one-to-one or many-to-one correspondence or a mixture. One-to-One Object Corresponder: Puts objects into 1-to-1 correspondence when it is clear the materials are a physical “pair.” In other situations, such as setting the table, may start to do 1-to-1, but then may keep on passing out items until they are all dispersed, or may skip some (due to the lack of clear matching, such as cups “near” plates). The child is sensitive to the relationships of “more than" and "less than” when working with very small numbers (from 1 to 2 years of age). Uses words to include "more," "less," or "same." Object Corresponder: Puts objects into one-to-one correspondence, although they may not fully understand that this creates equal groups.
2	3	Comparing Number: Perceptual Comparer	Compares collections that are quite different in number (e.g., one is at least twice the number as the other). Compares collections more similar in number but only for very small numbers (1, 2, and sometimes 3). Compares collections using number words “one” and “two."
3	3	Comparing Number: First-Second Ordinal Counter	Identifies the “first” and often “second” objects in a sequence.
3	3	Comparing Number: Early Comparer of Similar Items	Compares collections of 1 to 4 items verbally or nonverbally (“just by looking”). The items must be the same. May compare the small collections using number words “two” and “three” (approximately age 3; 2), and “three” and others (age 3; 6). Some do this even before they can accurately by using recognition of number/subitizing for these quantities. May transfer an ordering relation from one pair of collections to another.
3	4	Comparing Number: Early Comparer of Dissimilar Items	Matches small, equal collections consisting of different items, showing that they are the same number.
4	4	Comparing Number: Matching Comparer	Compares groups of 1–6 by matching.
4	4	Comparing Number: Counting Comparer (Same Size)	Accurately compares via counting, but only when objects are about the same size and groups are small (up to about 5). Not always accurate when the larger collection’s objects are smaller in size than the objects in the smaller collection. Accurately counts two equal collections, but, when asked, says the collection of larger blocks has more.

Trajectory	Levels		Level Description
5	5	Comparing Number: Spatial Extent Estimator—Small/Big	Estimates which set is more or less if the differences are clear (e.g., one is double the other). Names a "small number" (e.g. from 1-4) for sets that cover little space and a "big number" (10-20 or more) for sets that cover a lot of space. Children classify numbers "little"/"big" idiosyncratically, and this may change with the size of the to-be-estimated objects.
5	5	Comparing Number: Counting Comparer (5)	Compares with counting, even when larger collection's objects are smaller. Later, figures out how many more or less.
4	5	Comparing Number: Mental Number Line to 5	Uses knowledge of counting number relationships to determine relative size and position when given perceptual support.
4	5	Comparing Number: Serial Orderer to 5	Orders quantities (dots) or numerals up to 5. Similarly orders lengths marked into units.
5	5	Comparing Number: Ordinal Counter	Identifies and uses ordinal numbers from "first" to "tenth."
5	5	Comparing Number: Counting Comparer (10)	Compares with counting, even when larger collection's objects are smaller, up to 10.
5	5	Comparing Number: Mental Number Line to 10	Uses internal images and knowledge of number relationships to determine relative size and position.
6	6	Comparing Number: Serial Orderer to 6+	Orders quantities (dots) or numerals to 6 and beyond. Similarly orders lengths marked into units.
6	6	Comparing Number: Spatial Extent Estimator	Extends sets and number categories to include "small numbers," which are usually subitized, not estimated; "middle-size numbers" (e.g., 10–20); and "large numbers." The arrangement of the to-be-estimated set affects the difficulty.
7	7	Comparing Number: Place Value Comparer	Compares numbers with place value understandings.
7	7	Comparing Number: Mental Number Line to 100	Uses knowledge of number relationships and mental images, including how ones can be embedded in tens, to determine relative size and position.
7	7	Comparing Number: Scanning with Intuitive Quantification Estimator	Scans a group of objects and relates the results to a mental number line to perform a useful numerosity estimation.
7	8	Comparing Number: Mental Number Line to 1000	Uses internal images and knowledge of number relationships, including place value, to determine relative size and position.
8	8	Comparing Number: Benchmarks Estimator	Counts a portion of the to-be-estimated collection and uses that as a benchmark from which an estimate is made, intuitively or using repeated addition or multiplication. Scanning can be linked to recalled benchmarks.
8	8	Comparing Number: Composition Estimator	Decomposes or partitions the to-be-estimated set into convenient subset sizes, then recomposes the numerosity. Initially, this is done with regular arrangements using repeated addition or multiplication. Later, the process can be done with irregular arrangements and children more consistently use multiplication skills to recompose.

Trajectory Levels	Level Description
Adding / Subtracting	**Children learn that addition is the putting together of sets, here by counting (for example, the sum of 4 + 2 is the whole number that results from counting 2 numbers starting at 4) and subtraction is the separating of sets (counting backward).**

Age Range			
0	1	Adding & Subtracting: Arithmetic Senser:Foundations	Very young children are sensitive to combining or separating perceptual groups. An infant may observe, point, or make sounds while someone else introduces a quantity of objects. They will notice the effects of increasing or decreasing small collections by one item and may also be sensitive to the results of combining larger groups.
2	3	Adding & Subtracting: Preverbal +/-	Adds and subtracts very small collections (totals up to three), often making a collection rather than answering verbally.
4	4	Adding & Subtracting: Small Number +/-	Finds sums for Join, Result Unknown and Separate, Result Unknown problems with totals up to5 by “counting all” with objects.
4	5	Adding & Subtracting: Find Result +/-	Finds sums for Join, Result Unknown problems (“You had 3 apples and get 3 more, how many do you have in all?”) and part–part–whole (“There are 6 girls and 5 boys on the playground, how many children were there in all?”) problems by direct modeling, counting all, with objects. Solves take-away problems by separating with objects.
4	5	Adding & Subtracting: Make it N	Adds on objects to “make one number into another,” without needing to count from 1. Does not (necessarily) represent how many were added (this is not a requirement of this intermediate-difficulty problem type)
4	5	Adding & Subtracting: Find Change +/-	Finds the missing addend (e.g., 5 + _ = 7 or 9 - _ = 3) to solve Join and Separate, Change Unknown problems by adding on or taking away objects. Compares by matching in simple situations.
5	6	Adding & Subtracting: Counting Strategies +/-	Finds sums for joining (“You had 8 apples and get 3 more …”) and part–part–whole (“6 girls and 5 boys …”) problems with finger patterns and/or by counting on.
6	6	Adding & Subtracting: Part-Whole +/-	Has initial part–whole understanding and can solve all previous problem types using flexible strategies. May use some known combinations, such as 5 + 5 is 10. Sometimes can do “Start Unknown (e.g., _ + 6 = 11), but only by trial and error.
6	7	Adding & Subtracting: Numbers-in-Numbers +/-	Recognizes when a number is part of a whole and can keep the part and whole in mind simultaneously; solves “Start Unknown (e.g., _ + 4 = 9) problems with counting strategies.
6	7	Adding & Subtracting: Deriver +/-	Uses flexible strategies and derived combinations (e.g., “7 + 7 is 14, so 7 + 8 is 15) to solve all types of problems. Includes “Break Apart to Make Ten” (BAMT; explained in Chapter 6). Can simultaneously think of 3 numbers within a sum, and can move part of a number to another, aware of the increase in one and the decrease in another. May solve simple cases of multidigit addition (sometimes subtraction) by counting by tens and/or ones.
7	7	Adding & Subtracting: Problem Solver +/-	Solves all types of problems, with flexible strategies and known combinations. Multidigit may be solved by incrementing tens and ones by counting (latter not used for Join, Change Unknown).

Trajectory	Levels		Level Description
8	8	Adding & Subtracting: Multidigit +/-	Uses composition of tens and all previous strategies to solve multidigit +/− problems.

Trajectory Levels			Level Description
Composing Numbers			**Children learn another way of doing arithmetic by learning part-part-whole relationships ("seeing" 6 as 3 and 3, 5 and 1, or 0 and 6), which is a higher level of subitizing—conceptual subitizing.**
Age Range			
0	1	Composing Numbers: Actor on Parts: Foundations	Displays actions that show intuition about parts and wholes such as gathering objects together. Only nonverbally recognizes parts and wholes. Recognizes that sets can be combined in different orders but may not explicitly recognize that groups are additively composed of smaller groups.
1	3	Composing Numbers: Parts Combiner	Recognizes that sets can be combined in different orders, but may not explicitly recognize that groups are additively composed of smaller groups. The toddler also recognizes part-whole relations in nonverbal, intuitive, perceptual situations and can nonverbally represent parts that make a whole.
3	4	Composing Numbers: Inexact Part-Whole Recognizer	Knows that a whole is bigger than parts, but may not accurately quantify (label with numbers). (May show intuitive knowledge of commutativity, and, later, associativity with physical groups, later in more abstract contexts, including numbers.)
4	5	Composing Numbers: Composer to 4, then 5	Knows number combinations. Quickly names parts of any whole, or the whole given the parts.
5	6	Composing Numbers: Composer to 7	Knows number combinations to totals of 7. Quickly names parts of any whole, or the whole given parts. Doubles to 10.
5	6	Composing Numbers: Composer to 10	Knows number combinations to totals of 10. Quickly names parts of any whole, or the whole given parts. Doubles to 20.
7	7	Composing Numbers: Composer with Tens and Ones	Understands 2-digit numbers as tens and ones; count with dimes and pennies; 2-digit addition with regrouping.
6	7	Composing Numbers: Deriver +/-	Uses flexible strategies and derived combinations (e.g., “7 + 7 is 14, so 7 + 8 is 15) to solve all types of problems. Includes Break Apart to Make Ten (BAMT). Can simultaneously think of 3 numbers within a sum, and can move part of a number to another, aware of the increase in one and the decrease in another. Solves simple cases of multidigit addition (and, often, subtraction) by incrementing tens and/or ones.
7	7	Composing Numbers: Problem Solver +/-	Solves all types of problems, with flexible strategies and known combinations. Multidigit may be solved by incrementing or combining tens and ones (latter not used for join, change unknown).
7	8	Composing Numbers: Multidigit +/-	Uses composition of tens and all previous strategies to solve multidigit +/− problems.

Trajectory Levels			Level Description
Multiplying / Dividing			**Children learn that multiplying can be seen as the process of repeated addition and division as the splitting of a number into equal parts or groups.**
Age Range			
0	2	Multiplying/Dividing: Nonquantitative Sharer: Foundations	Gives some, but not necessarily an equal number to each person.
3	3	Multiplying/Dividing: Beginning Grouper and Distributive Sharer	Makes small groups (fewer than 5). Shares by “dealing out,” but usually only between 2 people. May not appreciate the numerical result.
4	5	Multiplying/Dividing: Grouper and Distributive Sharer	Makes small equal groups (fewer than 6). Deals out equally between two or more recipients, but may not understand that equal quantities are produced.
5	5	Multiplying/Dividing: Concrete Modeler x/÷	Solves small-number multiplying problems by grouping – making each group and counting all. Solves division/sharing problems with informal strategies, using concrete objects; up to 20 objects and 2-5 people. May not understand equivalence of groups.
6	6	Multiplying/Dividing: Parts and Whole X/÷	Understands the inverse relation between divisor and quotient in simple, concrete situations.
7	7	Multiplying/Dividing: Skip Counter X/÷	Uses repeated adding, additive doubling, or skip counting to solve multiplication and for measurement division (finding out how many groups) problems. Uses trial and error for partitive division (finding out how many in each group).
8	9	Multiplying/Dividing: Deriver X/÷	Uses strategies, patterns, de/composition (12 X 2 = 10 X 2 + 2 + 2) and derived combinations, such as multiplying X 9 as 10 – 1 or 7 X 8 from 7 X 7 + 7. Solves multidigit problems by operating on tens and ones separately.
8	9	Multiplying/Dividing: Problem Solver X/÷	Solves many types of multiplicative problems, with flexible strategies and known combinations. Multidigit may be solved using combinations separately on ones and tens.
8	9	Multiplying/Dividing: Partitive Divisor	Figures out how many are in each group. May first repeatedly add a divisor until the dividend is reached.

Trajectory Levels			Level Description

Fractions

Age Range

0	2	Fractions: Foundations: Early Proportional Thinker	Has an intuition about proportions
3	4	Fractions: Shape Equipartitioner	Can equipartition a whole shape, such as a circle or rectangle.
4	5	Fractions: Half Recognizer	Recognizes "halves" at least in continuous (e.g., area) representations, especially in the context of fair shares. Recognition of the need for ½ when sharing an odd number of objects. Intuitively and visually, combines regions that are a part of a whole, showing initial foundations for addition.
4	6	Fractions: Unit Fraction Recognizer	Recognizes unit fractions in simple discrete (countable) and maybe continuous (e.g., area) representations for ½, 1/3, and ¼ and understands intuitively that they are formed by dividing a whole into equal parts. Names these shares.
7	7	Fractions: Fraction Recognizer	Recognizes simple (small number denominators) fractions in familiar continuous and discrete contexts.
7	7	Fractions: Fraction Maker from Units	Creates a fraction representation with equal parts and the correct number of repetitions of a unit fraction. Labels that fraction with written fraction notation. Compares fraction representations and states which is the larger number.
8	8	Fractions: Fraction Maker	Creates a fraction representation with equal parts and the correct number of repetitions of a unit or nonunit fraction (as long as they are not greater than a whole—fraction may be only a "part-of-a-whole"). Compare simple common fractions using physical models.
8	8	Fractions: Fraction Repeater	Creates fraction representations with repetitions of unit and non-unit fractions, including results greater than a whole. Moves beyond fraction as a "part-of-a-whole" to fraction as a number with a relation to the reference whole. Understands that fractions with the same denominator can be added or subtracted using the units of the unit fraction. Compares simple common fractions using models such as the number line—understands that two factions are equal when they represent the same portion of a whole or have the same length on the number line.
8	8	Fractions: Fraction Arithmetic +/-	Adds and subtracts simple common fractions using physical models.
8	8	Fractions: Fraction Arithmetic x/÷	Multiples simple common fractions using rectangle array model.
8	8	Fractions: Fraction and Integer Sequencer	Represents of simple ratios as percentages, fractions and decimals. Orders integers, positive fractions, and decimals.

Trajectory Levels			Level Description
Patterning			**Children learn to find and use mathematical regularities and structures. Children learn to identify, duplicate, and extend sequential patterns such as ABCABCABC, but also to find regularities and structures in number and geometry. Repeating patterning skills predicts later math knowledge even after controlling for prior math knowledge. It is a critical math topic!**
Age Range			
0	2	Patterning: Intuitive Patterner: Foundations	Detects and uses patterning implicitly and intuitively, such as in movement activities or common nursery rhymes that repeat words and action. May be attentive to repeating patterns without recognizing them explicitly or accurately, often attending to individual attributes such as color.
2	3	Patterning: Pattern Recognizer	Recognizes a simple pattern, usually ABABAB, as a pattern, even if doesn't yet name or describe it.
3	4	Patterning: Patterner AB	Recognizes, describes, and builds repeating ABAB patterns. These involve the following, which many children learn in this order, although this can vary by the task.1 Fixes AB: Fills in missing element of an ABAB pattern. Duplicates AB: Duplicates ABABAB pattern (at first may have to work close to the model pattern, but eventually can build the same pattern away from the model pattern or when the model is out of sight). Extends AB: Extends AB patterns to add multiple units to the end of the pattern. This is easier for children if the pattern ends with a complete unit, but they eventually learn to extend those that end with a partial unit.
4	5	Patterning: Patterner	Recognizes, describes, and builds repeating patterns, including AB but also patterns with core units such as AAB, ABC, and AABC.
4	5	Patterning: Pattern Translator and Unit Recognizer	Translates patterns into new media or using new materials; that is, abstract and generalize the pattern. Identifies the smallest core unit of a repeating pattern. (Most research indicates this develops later, Miller et al., 2016.) In functional thinking situations (e.g., p. #), does not yet see math relationships in sets of data, level 1.
5	7	Patterning: Numeric Patterner	Describes a pattern numerically, can translate between geometric and numeric representation of a series. In functional thinking, builds and perceives a t-chart as a sequence of particular instances, level 2.
5	7	Patterning: Beginning Arithmetic Patterner	Recognizes and uses arithmetic patterns with perceptual or pedagogical support, often first those that involve properties of zero. The child also accepts number sentences not in the form of 3 + 4 = 7 (e.g., 7 = 3 + 4, or even 3 + 4 = 2 + 5). This represents a move from an "equals-as-an answer" notion to recognizing that equals means equivalent numbers. In functional thinking, builds two sets (e.g., in a t-chart) following two separate general rules, level 3.
6	7	Patterning: Relational Thinker +/-	Recognizes and uses patterns that involve addition and subtraction and, understanding equality, can compare two sides of a number sentence with reasoning, and thus does not have to carry out computations. In functional thinking, creates functional relationships between two data sets but only for specific cases (Blanton et al., 2015)level 4. May use letters to represent numbers, but only as representing objects or fixed values.

6	7	Patterning: Relational Thinker-Symbolic +/-	Recognizes and uses patterns that involve addition and subtraction and anunderstanding of equality. Can compare two sides of a number sentence with reasoning, even when the quantities are represented by variables, such as a + b = b + a. . In functional thinking, generalizes functional relationships between two data sets, at first just noticing, and later a quantitative relationship (Blanton et al., 2015). level 5, 6 Uses letters for unknown numbers, an initial algebraic notion.
6	8	Patterning: Relational Thinker with Multiplication	Recognizes and uses patterns that involve multiplication as repeated addition and use of the distributive property to partition number facts. In functional thinking, generalizes functional relationships between two data sets, (Blanton et al., 2015). level 7. Uses letters as variables represent this relationhip≥
6	8	Patterning: Functional Algebraic Thinker	In functional thinking, generalizes functional relationships between two data sets, understanding the boundaries of generalizability and thus understanding the function as a math object (Blanton et al., 2015). level 8.

Trajectory Levels			Level Description
2D Shapes			**Children learn to match, name, describe, represent, construct, and classify two-dimensional ("flat") geometric shapes, increasingly using mathematical properties of the shapes.**
Age Range			
0	2	Shapes: "Same Thing" Comparer: Foundations	Compares real-world objects
0	2	Shapes: Shape Matcher-Identical, Sizes, & Orientations	Matches familiar shapes (circle, square, typical triangle) with same size and orientation. Matches familiar shapes with different sizes. Matches familiar shapes with different orientations.
3	3	Shapes: Shape Recognizer-Typical	Recognizes and names a typical circle, square, and, less often, triangle. May physically rotate shapes in atypical orientations to mentally match them to a prototype.
3	4	Shapes: Shape Matcher—More Shapes, Sizes, & Orientations, Combinations	Matches a wider variety of shapes with same size and orientation. Matches a wider variety of shapes with different sizes and orientations. Matches combinations of shapes to each other.
4	4	Shapes: Shape Recognizer—Circles, Squares, and Triangles	Recognizes some less typical squares and triangles and may recognize some rectangles, but usually not rhombuses (diamonds). Often doesn't differentiate sides/corners. (duplicate for Rectangles)
4	4	Shapes: Constructor of Shapes from Parts—Looks Like	Uses manipulatives representing parts of shapes, such as sides, to make a shape that "looks like" a goal shape. May think of angles as a corner (which is "pointy").
4	5	Shapes: Shape Recognizer-All Rectangles	Recognizes rectangles of all sizes, shapes, and orientations.
4	5	Shapes: Side Recognizer	Identifies sides as distinct geometric objects with attributes.
4	5	Shapes: Most Attributes Comparer	Looks for differences in attributes, examining full shapes, but may ignore some spatial relationships.
4	5	Shapes: Corner (Vertex, Angle) Recognizer	Recognizes angles as separate geometric objects, at least in the limited context of "corners."
5	5	Shapes: Shape Recognizer—More Shapes	Recognizes most familiar shapes and typical examples of other shapes, such as hexagon, rhombus (diamond), and trapezoid.
6	6	Shapes: Shape Identifier	Names most common shapes, including, for example, rhombuses, hexagons, octagons, and trapezoids, without making mistakes, such as calling ovals "circles." Recognizes (at least) right angles, so distinguishes between a rectangle and a parallelogram without right angles.

Trajectory	Levels		Level Description
7	7	Shapes: Angle Recognizer—More Contexts	Recognizes and describe contexts in which angle knowledge is relevant, including corners (can discuss "sharper" angles), crossings (e.g., a pair of scissors), and, later, bent objects and bends (sometimes bends in paths and slopes). Only later can explicitly understand how angle concepts relate to these contexts (e.g., initially may not think of bends in roads as angles; may not be able to add horizontal or vertical to complete the angle in slope contexts; may even see corners as more or less "sharp" without representing the lines that constitute them). Often does not relate these contexts and may represent only some features of angles in each (e.g., oblique line for a ramp in a slope context).
7	7	Shapes: Parts of Shapes Identifier	Identifies shapes in terms of their components.
7	7	Shapes: Congruence Superposer	Moves and places objects on top of each other to determine congruence. Can also determine congruence by comparing all attributes and all spatial relationships.
7	7	Shapes: Constructor of Shapes From Parts - Exact	Uses manipulatives representing parts of shapes, such as sides and angle "connectors," to make a shape that is completely correct, based on knowledge of components and properties — relationships between the components.
8	8	Shapes: Angle Representer	Represents various angle contexts as two lines, explicitly including the reference line (horizontal or vertical for slope; a "line of sight" for turn contexts) and, at least implicitly, the size of the angle as the rotation between these lines (may still maintain misconceptions about angle measure, such as relating angle size to the length of side's distance between endpoints, and may not apply these understandings to multiple contexts).
8	8	Shapes: Congruence Representer	Refers to geometric properties and explains with transformations.
8	8	Shapes: Shape Class Identifier	Uses class membership (e.g., to sort), not explicitly based on properties.
8	8	Shapes: Shape Property Identifier	Uses properties explicitly. Can see the invariants in the changes of state or shape, but maintaining the shapes' properties.
8	8	Shapes: Property Class Identifier	Uses class membership for shapes (e.g., to sort or consider shapes "similar") explicitly based on properties, including angle measure. Is aware of restrictions of transformations and also of the definitions and can integrate the two. Sorts hierarchically, based on properties.
8	8	Shapes: Angle Synthesizer	Combines various meanings of angle (turn, corner, slant), including angle measure.

Trajectory Levels	Level Description
Composing 2D Shapes	**Children learn to compose (put together) two-dimensional shapes to make other shapes, solving puzzles or making pictures or designs, or decompose ("take apart") shapes into parts (simpler shapes).**

Age Range

0	3	Composing 2D Shapes: Separate Shapes Actor: Foundations	Infants and toddlers manipulate shapes as individuals, but usually do not combine them to compose a larger shape.
4	4	Composing 2D Shapes: Piece Assembler	Makes pictures in which each shape represents a unique role (e.g., one shape for each body part) and shapes touch. Fills simple puzzles in which all shapes are outlined, often using trial and error.
4	5	Composing 2D Shapes: Picture Maker	Puts several shapes together to make one part of a picture (e.g., two shapes for one arm). Uses trial and error, and does not anticipate creation of new geometric shape. Chooses shapes using “general shape” or side length. Fills “easy” “Pattern Block Puzzles” that suggest the placement of each shape (note that in the example on the right the child is trying to put a square in the puzzle where its right angles will not fit).
4	5	Composing 2D Shapes: Simple Decomposer	Decomposes (“takes apart” into smaller shapes) simple shapes that have obvious clues as to their decomposition.
4	5	Composing 2D Shapes: Shape Composer	Composes shapes with anticipation (“I know what will fit!”). Chooses shapes using angles as well as side lengths. Rotation and flipping are used intentionally to select and place shapes. In the “Pattern Block Puzzles” below, all angles are correct, and patterning is evident.
5	6	Composing 2D Shapes: Substitution Composer	Makes new shapes out of smaller shapes, and uses trial and error to substitute groups of shapes for other shapes to create new shapes in different ways.
6	6	Composing 2D Shapes: Shape Decomposer (with Help)	Decomposes shapes using imagery that is suggested and supported by the task or environment.
6	7	Composing 2D Shapes: Shape Composite Repeater	Constructs and duplicates units of units (shapes made from other shapes) intentionally; understands each as being both multiple small shapes and one larger shape. May continue a pattern of shapes that leads to tiling.
7	7	Composing 2D Shapes: Shape Decomposer with Imagery	Decomposes shapes flexibly using independently generated imagery. That is, decomposition is intentionally specified by the child.
8	8	Composing 2D Shapes: Shape Composer-Units of Units	Builds and applies units of units (shapes made from other shapes). For example, in constructing spatial patterns, extends patterning activity to create a tiling with a new unit shape—a unit of unit shapes that they recognize and consciously construct.
8	8	Composing 2D Shapes: Shape Decomposer with Units of Units	Decomposes shapes flexibly using independently generated imagery and planned decompositions of shapes that themselves are decompositions.

Trajectory Levels			Level Description
Disembedding Shapes			**Children learn to identify structures within embedded figures (finding "hidden shapes" within more complex diagrams).**
Age Range			
3	3	Disembedding shapes: Intuitive Disembedder: Foundations	Can remember and reproduce only one or a small collection of non-overlapping (isolated) shapes.
4	4	Disembedding shapes: Simple Disembedder	Identifies frame of complex figure. Finds some shapes in arrangements in which figures overlap, but not in those in which figures are embedded within others.
5	6	Disembedding shapes: Shapes in Shapes Disembedder	Identifies shapes embedded within other shapes, such as concentric circles and/or a circle in a square. Identifies primary structures in complex figures.
7	7	Disembedding shapes: Secondary Structure Disembedder	Identifies embedded figures even when they do not coincide with any primary structures of the complex figure.
8	8	Disembedding shapes: Complete Disembedder	Successfully identifies all varieties of complex arrangements.

Trajectory Levels			Level Description
3D Shapes			
Age Range			
0	2	3D Perceiver: Foundations	Children can perceive 3D shapes accurately from infancy, however, this competence is limited to continuously moving objects, rather than single or even multiple static views of the same object.
3	4	3D Prototype Recognizer	Children can recognize some prototypical 3D shapes, such as the sphere and cube, using formal or informal names. However, may use 2D vocabulary to name some 3D shapes and describe solids using a variety of informal characteristics, such as "poinyness" or "slenderness."
5	6	3D Shape Recognizer	Recognizes more 3D shapes (solids), using informal and some formal names). Recognizes faces as 2D shapes.
7	7	3D Face Counter	Recognizes all faces of a solid as 2D shapes, counting faces accurately.
8	8	3D Shape Identifier	Identifies most solids, naming several of their attributes. Can identify the common solid created by a particular net.
8	8	3D Shape Class Identifier	Identifies most solids, based on their properties.

Trajectory Levels	Level Description
Composing 3D shapes	**Children learn to compose (put together) three-dimensional shapes to make other 3D shapes.**

Age Range

0	1	Composing 3D shapes: Separate Blocks Actor: Foundations	Either places blocks randomly or manipulates shapes as individuals, but does not combine them to compose a larger shape. May pound, clap together, or use slide blocks or single blocks to represent an object, such as a house or truck.
1	1	Composing 3D shapes: Stacker	Shows use of the spatial relationship of “on” to stack blocks, although choice of blocks may be unsystematic.
1	1	Composing 3D shapes: Line Maker	Shows use of relationship of “next to” to make a (one-dimensional) line of blocks.
2	2	Composing 3D shapes: Same Shape Stacker	Shows use of relationship of “on” to stack congruent blocks, or those that show a similarly helpful relationship to make stacks or lines.
2	3	Composing 3D shapes: Piece Assembler (3D)	Builds vertical and horizontal components within a building, but within a limited range, such as building a “floor” or a simple “wall.” These, then, are two-dimensional structures.
3	4	Composing 3D shapes: Picture Maker (3D)	Uses multiple spatial relations, extending in multiple directions and with multiple points of contact among components, showing flexibility in integrating parts of the structure. Produces arches, enclosures, corners, and crosses, although may use unsystematic trial and error and simple addition of pieces.
4	5	Composing 3D shapes: Shape Composer (3D)	Composes shapes with anticipation, understanding what 3D shape will be produced with a composition of 2 or ore other (simple, familiar) 3D shapes. Can produce arches (with vertical interior space), enclosures (with internal horizontal space), corners, and crosses systematically. Builds enclosures and arches several blocks high. Later in this level, children add depth to make 3D structures, and they add roofs across structures multiple blocks high (but they may have no internal spaces).
5	6	Composing 3D shapes: Substitution Composer and Shape Composite Repeater (3D)	Substitutes a composite for a congruent whole. Builds complex bridges with multiple arches, ramps and stairs at the ends. Structures are 3D, often including roofs and multiple internal spaces.
6	8	Composing 3D shapes: Shape Composer - Unit of Units (3D)	Makes complex towers or other structures, involving multiple levels with ceilings (fitting the ceilings), and adult-like structures with blocks, including arches and other substructures.

Trajectory Levels	Level Description
Spatial Visualization	**Children develop the processes of generating, maintaining, and manipulating mental images of two- and three-dimensional objects, including moving, matching, and combining them. It is a critical math topic!**

Age Range			
0	1	Spatial Visualization: Intuitive Mover: Foundations	Explores the size and shape of objects by observing them as they move in space, using trial and error to discover how they fit into space, and eventually predicting what will fit inside a space without attempting all possible solutions. Such skills will eventually support future spatial visualization.
1	2	Spatial Visualization: Concrete Slider, Flipper, Turner	Can move shapes to a location by physical trial and error.
3	4	Spatial Visualization: Simple Slider and Turner	Slides and turns objects accurately in easy tasks, guided by an early intuition that starts the motion and then adjusts (the motion, direction, or amount) in real time as the motion is carried out.
5	5	Spatial Visualization: Beginning Slider, Flipper, Turner	Uses the correct motions guided by more developed intuition, but not always accurate in direction and amount (adjusts these with trial and error). Knows a shape has to be flipped to match another shape, but flips it in the wrong direction.
6	6	Spatial Visualization: Slider, Flipper, Turner	Performs slides and flips, often only horizontal and vertical, using manipulatives but guided by mental images of these motions (of turns of 45, 90, and 180° and flips over vertical and horizontal lines). That is, they can mentally imagine the motion and the result of it. Knows a shape must be turned 90° to the right to fit into a puzzle.
7	7	Spatial Visualization: Diagonal Mover	Performs diagonal slides and flips as well as all motions from previous levels. Knows a shape must be turned flipped over an oblique line (45° orientation) to fit into a puzzle.
8	8	Spatial Visualization: Mental Mover	Predicts results of moving shapes using mental images (any direction or amount).

Trajectory Levels	Level Description
Spatial Orientation	**Children learn the relationships between different positions in space (includes maps and coordinates).**

Age Range

0	0	Spatial Orientation: Foundations of Spatial Orientation	Uses the earliest of two types of cognitive systems for spatial orientation—knowing where you are and how to get around in the world. 1. Response Learning: Uses the first self-based system – that is, related to the child's own position and movements. Notes a pattern of movements that have been associated with a goal. 2. Cue Learning: Uses the first external-based systems, based on familiar landmarks.
0	2	Spatial Orientation: Path Integrater	Remembers and can repeat movements they have made including the approximate distances and directions.
1	2	Spatial Orientation: Place Learner	Creates "mental maps" by storing locations, distances, and directions to landmarks and solves spatial problems. Uses the walls of a room as a frame of reference; uses spatial vocabulary, such as "in," "on," and "under," along with vertical directionality terms as "up" and "down."
2	3	Spatial Orientation: Local-Self Framework User	Uses distant landmarks to find objects or location near them, even after they have moved themselves relative to the landmarks, if the target object is specified ahead of time. Orients a horizontal or vertical line in space (Rosser, Horan, Mattson, & Mazzeo, 1984). Uses spatial vocabulary to direct attention to spatial relations, including more difficult terms such as "beside" and "between."
4	4	Spatial Orientation: Small Local Framework User	Locates objects after movement, even if target is not specified ahead of time. Searches a small area comprehensively, often using a circular search pattern. Uses words referring to frames of reference such as "in front of" and "behind" or "left" and "right." In meaningful graphing contexts, extrapolates lines from positions on both axes (like a coordinate grid) and determines where they intersect.
5	5	Spatial Orientation: Local Framework User	Locates objects after moving, maintaining the overall shape of the arrangement of objects. Represents objects' positions relative to landmarks (e.g., about halfway in between two landmarks) and keeps track of own location in open areas or mazes. Uses spatial vocabulary to direct attention to spatial relations. Uses coordinate labels in simple situations such as games.
6	6	Spatial Orientation: Map User	Locates objects using maps with pictorial cues. Extrapolates (extends) two coordinates, understanding the integration of them to one position, as well as use coordinate labels in simple situations.
7	7	Spatial Orientation: Coordinate Plotter	Reads and plots coordinates on maps.
8	8	Spatial Orientation: Route Map Follower	Follows a simple route map, with more accurate direction and distances.
8	8	Spatial Orientation: Framework User	Uses general frameworks that include the observer and landmarks. May not use precise measurement even when that would be helpful, unless guided to do so. Can follow and create maps, even if spatial relations are transformed.

Trajectory Levels			Level Description
Length			**Children learn to compare and then quantify how far it is between the endpoints of an object.**
Age Range			
1	2	Length: Length Senser: Foundations	Makes simple comparisons of length intuitively (similar to what we saw in Subitizing) as young as six months of age. However, may not recognize length as a distinct a
3	3	Length: Length Quantity Recognizer	Identifies length/distance as an attribute. May understand length as an absolute descriptor (e.g., all adults are tall), but not as a comparative (e.g., one person is taller than another). May compare non-corresponding parts of shape in determining side length.
4	4	Length: Length Direct Comparer	Physically aligns two objects to determine which is longer or if they are the same length. Uses terms: long, longer, longest.
4	5	Length: Length Indirect Comparer	Compares the length of two objects by representing them with a third object. Uses terms: long, longer, longest, short, shorter, shortest. When asked to measure, may assign a length by guessing or moving along a length while counting (without equal-length units). May be able to measure with a ruler, but often lacks understanding or skill (e.g., ignores starting point).
4	5	Length: Serial Orderer to 5	Orders lengths, marked in 1 to 5 units. Also, can compare unmarked lengths that are clearly different using broad categories ("big" and "small") and so can order 3 to 5 such objects but only by trial-and-error. With an increase in working memory, begins to build a mental image of the final ordering in which the lengths increase "bit by bit" with each successive length the smallest increase. This leads to more accurate and somewhat more efficient ordering. (This level develops in parallel with "End-to-End Length Measurer".)
4	5	Length: End-to-End Length Measurer	Lays units end to end. May not recognize the need for equal-length units or be able to measure if there are fewer units that needed. The ability to apply resulting measures to comparison situations develops later in this level. (This develops in parallel with "Serial Orderer to 5 (Length)").
5	6	Length: Serial Orderer to 6+	Orders lengths, marked in 1 to 6 units. Understands at least intuitively that any set of objects of different lengths can be placed into a series that always increases (or decreases) in length, so spontaneously seriates with few errors by selecting the shortest (or longest) object, then the next shortest (the one with the "smallest difference"), and so forth.
7	7	Length: Length Unit Relater and Repeater	Measures by repeating (iterating) a single unit and understands the need for equal-length unit. Relates the size and number of units (inverse relationship). Can add two lengths to obtain the length of a whole. Often can use rulers with minimal guidance in straightforward situations.
8	8	Length: Length Measurer	Considers the length of a bent path as the sum of its parts (not the distance between the endpoints). Measures, knowing need for identical units, relationship between different units, partitions of unit, zero point on rulers, and accumulation of distance. Begins to estimate.

Trajectory	Levels		Level Description
8	8	Length: Conceptual Ruler Measurer	Possesses an “internal” measurement tool. Mentally moves along an object, segmenting it and counting the segments. Operates arithmetically on measures (“connected lengths”). Subdivides a unit at least into halves. Estimates with accuracy.

Area — **Children learn to compare and then quantify an amount of two-dimensional surface that is contained within a boundary.**

Age Range		Trajectory Level	Level Description
0	3	Area: Area Senser: Foundations	Even children in their first year are sensitive to area. However, may not explicitly recognize area as an attribute (separate from general size, such as “small” and “big”) for some time. If asked to fill in a rectangle, preschoolers may just draw approximations of circles (Mulligan, Prescott, Mitchelmore, & Outhred, 2005). Uses side matching strategies in comparing areas.
4	4	Area: Area Quantity Recognizer	Perceives the amount of two-dimensional space and can make intuitive comparisons. However, when asked to compare, may compare lengths more than areas because lengths are salient and familiar to them (e.g., compare one side of one piece of paper to the side of another) or make estimates based on a “length plus (not times) width” intuition. However, may compare areas correctly if the task suggestions superposition (putting one on top of the other). Asked to partition a space into squares or copy an image of a rectangle partitioned into an array (rows and columns), may simply draw squares (usually!) inside the rectangle or other types of shapes or short paths on or around the rectangle.
4	5	Area: Physical Coverer and Counter	Prompted to measure, attempts to cover a rectangular space with physical tiles. However, doesn't organize or structure the 2D space without considerable perceptual support, such as a grid that outlines each individual unit. In drawing (or imagining and pointing to count squares as units of area), represents only certain aspects of that structure, such as approximately rectangular shapes next to one another. Makes comparison areas based on simple, direct comparisons (e.g., a child places one sheet of paper over another piece of paper to select the sheet that covers more space).
5	5	Area: Complete Coverer and Counter	Draws a complete covering of a specific region without gaps or overlaps and in approximations of rows. When provided with more than the total number of physical tiles needed, can build a region of specified area (e.g., build a rectangle with an area of 12 from a pile of 20 tiles).
5	5	Area: Area Unit Relater and Repeater	Counts individual units, often trying to use the structure of rows. To cover a region with physical units, repeats (iterates) an individual unit. Draws a complete covering based on an intuitive notion of rows and columns, making equal sized units, but often draws them one at a time. That is, draws individual, mainly equal-sized units that are lined up but may not see groups of units making up individual rows or columns. Relates the size and number of units to cover a region, recognizing that differently sized units will result in different measures and that the larger the unit, the fewer will be needed. Compares areas by accurately counting units in each and comparing the resulting measures.

Trajectory	Levels		Level Description
6	6	Area: Initial Composite Structurer	Identifies a square unit as both a unit and a component of a larger unit of units (a row, column, or group) and uses those structures in counting or drawing. However, needs figural support to structure the space themselves (this may include physical motions of some of the tiles or drawing some collections of units rather than from using the dimensions). At this level, usually does not coordinate the width and height and in measuring, may not use the dimensions of the rectangle to constrain the unit size. Makes reasonable estimates of areas.
7	7	Area: Area Row and Column Structurer	Decomposes and recomposes partial units to make whole units. For example, draws rows as rows making parallel horizontal lines and so forth. Begins conserving area and reasons about additive composition of areas (e.g., how regions that look different can have the same area measure) and recognizes the need for space-filling in most contexts.
8	8	Area: Array Structurer	With linear measures or other similar indications of the two dimensions, multiplicatively iterates rows or columns to determine the area. Does not need to draw in the array to do so. Has an abstract understanding of the rectangular area formula. Understands and justifies that differently-shaped regions can have the same areas. Compares regions with transitive reasoning (e.g., A is greater than B, B is greater than C, so I know A is greater than C).

Trajectory Levels	Level Description
Volume	**Children learn to compare and then quantify an amount that can be "packed" or "filled" into a 3D space with iterations of a fluid unit that takes the shape of the container.**

Age Range

0	2	Volume: Volume Senser: Foundations	Sensitive to volume even in the first year; however, they may not for some time explicitly recognize volume as an attribute (separate from general size, such as “small” and “big”).
1	3	Volume: Volume Quantity Recognizer	Identifies capacity or volume as attribute. Builds with blocks, associating more blocks with terms like “big” and fewer blocks with terms like “small.”
3	5	Volume: Volume Filler	Can compare two containers by pouring one into the other (although can be confused at “which holds more” at first). Fills a container using another (smaller container) and counts the number needed to completely fill the larger container (but may not use accurately filled scoops and may not focus on quantifying the total volume or capacity). In packing situations, places cubes into a rectangular box to fill it. Eventually packs entire box with cubes in an organized way. Compares objects by physically or mentally aligning; refers to at least two dimensions of objects. May be able to compare two containers using a third container and transitive reasoning.
5	6	Volume: Volume Quantifier	Partial understanding of cubes as filling a space. Able to estimate number of scoops needed to fill. Able to attend to both the portion of container filled and the portion remaining unfilled. Recognizes when container is half full. Exhibits initial spatial structuring. Packs box neatly and completely with cubes; may count one cube at a time, while packing, to determine total. Compares objects by physically or mentally aligning and explicitly recognizing three dimensions.
7	7	Volume: Volume Unit Relater and Repeater	Uses simple units to fill containers, with accurate counting. Relates size and number of units explicitly; understands that fewer larger than smaller units will be needed to fill or pack a given container. Can accurately convert units in 1:2 ratio.
7	7	Volume: Initial Composite 3-D Structurer	Understands cubes as filling a space but does not use layers or multiplicative thinking. Moves to more accurate counting strategies. Relates number of cubes to cubic units as measured by capacity. Given a graduated cylinder marked in cubic-inch units, child understands that sand filled to the 10 in the cylinder would fill a box that holds ten, 1-inch cubes. Begins to visualize and operate on composite units such as rows or columns (what we call a 1x1xn core). Iterates to pack the space completely, accounting for “internal/ hidden” cubes. Decomposes space, allowing for accurate use of units and subunits. Recognizes when a box is half full, visualizes remaining rows or columns.

8	8	Volume: 3-D Row and Column Structurer	Able to coordinate flexibly filling, packing, building aspects of volume. Shows a propensity for additive comparisons (e.g., “this one has 12 more”) but may show some nascent multiplicative comparisons (e.g., “this one is four times as big”). Initially counts or computes (e.g., number of rows times number of columns) the number of cubes in one layer, and then uses addition or skip counting by layers to determine the total volume. Eventually moves to multiplication (e.g., number of cubes in a layer times number of layers).
8	8	Volume: 3-D Array Structurer	Has an abstract understanding of the rectangular prism volume formula. Shows a propensity for multiplicative comparisons, coordinates multiplicative and additive comparisons flexibly. With linear measures or other similar indications of the three dimensions, multiplicatively iterates cubes in a row, column, and/or layers to determine the area. Constructions and drawings are not necessary. In multiple contexts, children can compute the volume of rectangular prisms from its dimensions and explain how that multiplication creates a measure of volume.

Trajectory Levels			Level Description
Angle and Turn			**Children learn to compare and then quantify angle and turn measure.**
Age Range			
1	2	Angle and Turn measurement: Angle and Turn Senser: Foundations	Infants are sensitive to angles-as-turning, both turning objects and their own body. See more at the first three levels of the Spatial Orientation and the first level of Spatial Visualization.
2	3	Angle and Turn measurement: Intuitive Angle Builder	Intuitively uses some angle measure notions in everyday settings, such as building with blocks, solving puzzles, and walking.
4	5	Angle and Turn measurement: Implicit Angle User	Uses angles and, at least implicitly, some angle measure concepts, such as parallelism and perpendicularity—in physical alignment tasks, construction with blocks, or other everyday contexts. May identify corresponding angles of a pair of congruent triangles using physical models. Uses the word “angle” or other descriptive vocabulary to describe some of these situations.
6	6	Angle and Turn measurement: Angle Matcher	Matches angles concretely. Explicitly recognizes parallels from non-parallels in specific contexts (Mitchelmore, 1992). Sorts angles into “smaller” or “larger” (but may be misled by irrelevant features, such as length of line segments).
7	7	Angle and Turn measurement: Angle Size Comparer	Differentiates angle and angle size from shapes and contexts, and compares angle sizes. Recognizes right angles, and then equal angles of other measures, in different orientations. Compares simple turns. (Note that, without instruction, this and higher levels may not be achieved even by the end of the elementary grades.)
8	8	Angle and Turn measurement: Angle Measurer	Understands angle and angle measure in both primary aspects, and can represent multiple contexts in terms of the standard, generalizable concepts and procedures of angle and angle measure (e.g., two rays, the common endpoint, rotation of one ray to the other around that endpoint, and measure of that rotation).

Trajectory Levels	Level Description
Classification and Data Analysis	**This learning trajectory develops children's ability to understand, gather, and use data. In the early years, data is an important context for solving problems but data analysis itself develops slowly. Children first learn the foundational ideas and processes of classification, and then learn to quantify the categories, that is, tell how many in each group. Eventually children learn to gather data to answer a question or make a decision is an effective means to develop applied problem solving and number and/or spatial sense, as children simultaneously learn about data representations and analyses.**

Age Range

0	1	Classification & Data: Foundations: Similarity Intuiter	Intuitively recognizes objects or situations as similar in some way (objects to suck or not, 2 weeks). Places objects together that are different (6 months) and then alike (12 months).
1	2	Classification & Data: Similar/Dissimilar Maker	By 18 months, forms sets in which objects in each set are identical and objects in the other sets are different, and by 2 years, intuitively forms groups with objects that are similar on some attributes (may be mixed and inconsistent), but not necessarily identical.
3	3	Classification & Data: Simple Sorter	Follows verbal rules for sorting scaffolded by an adult. (These may be made with shifting criteria; nevertheless, they play an essential role in number, through the unitizing process.) Can “fix” a simple sort with mistakes.
4	4	Classification & Data: Sorter by Similar Attributes	Sorts objects according to an explicit attribute (although still may decide to switch attributes during the sorting). The end result may appear to reflect adult categorizations, but often has a different basis, such as general resemblance.
4	6	Classification & Data: Consistent, Flexible Sorter	Sorts consistently by a single attribute and re-classifies by different attributes. Sorts consistently and exhaustively by an attribute, given or created, and uses the terms “some” and “all.”
4	6	Classification & Data: Data Case Viewer	Associates a value with an individual case. Uses numeric data to identify largest/smallest cases. Before this level, children may be “pointers” in which data records point to the entire event (“We talked about favorite colors”). They use it like string tied around a finger, to remember that they did something.
5	6	Classification & Data: Data Classifier	Data Treats cases with similar values as the same. Uses to compare category frequencies (most and least popular case-types). Visually compares two graphs.
5	7	Classification & Data: Multiple Attribute Classifier	Classifies objects by multiple attributes in a single sort.

Trajectory	Levels		Level Description
7	8	Classification & Data: Data Aggregator	Classifies objects that may be perceptually different by more abstract attributes such as function or conceptual attributes. Focuses on features of the data set as a whole. Uses to describe relative frequency and density (shape), and location (centers). Begins to understand the concepts of expectation (averages and probabilities) and variation ("spread" of values). Views ranges in data or view the mode (the number or range of numbers that occurs most frequently). Eventually, can focus on features of the data set as a whole, including the relative frequencies, density ("shape"), and location (centers, such as the mean).
8	8	Classification & Data: Hierarchical Classifier	Classifies categories and subcategories using hierarchical inclusion. Conscientiously classifies according to multiple attributes, naming and relating the attributes, understanding that objects could belong to more than one group.
8	8	Classification & Data: Data Representer	Shows an appreciation of the "center" of graphs and for their variation or spread. Compares graphs of data sets of the same size accurately.